THE EVOLUTION OF...

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Exceptional Objects

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1. INTRODUCTION. In the mind of every mathematician, there is tension between the general rule and exceptional cases. Our conscience tells us we should strive for general theorems, yet we are fascinated and seduced by beautiful exceptions. We can't help loving the regular polyhedra, for example, but arbitrary polyhedra leave us cold. (Apart from the Euler polyhedron formula, what theorem do you know about *all* polyhedra?) The dream solution to this dilemma would be to find a general theory of exceptions—a complete description of their structure and relations—but of course it is still only a dream. A more feasible project than mathematical unification of the exceptional objects is historical unification: a description of some (conveniently chosen) objects, their evolution, and the way they influenced each other and the development of mathematics as a whole.

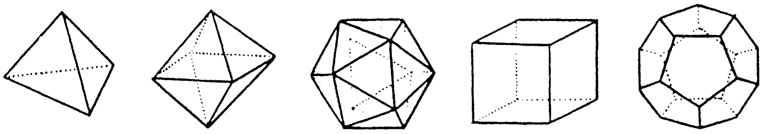
It so happens that patterns in the world of exceptional objects have often been discovered through an awareness of history, so a historical perspective is worthwhile even for experts. For the rest of us, it gives an easy armchair tour of a world that is otherwise hard to approach. A rigorous understanding of the exceptional Lie groups and the sporadic simple groups, for example, might take a lifetime. They are some of the least accessible objects in mathematics, and from most viewpoints they are way over the horizon. The historical perspective at least puts these objects in the picture, and it also shows lines that lead naturally to them, thus paving the way for a deeper understanding. I hope to show that the exceptional objects do have a certain unity and generality, but at the same time they are important *because* they are exceptional.

Some of the ideas in this article arose from discussions with David Young, in connection with his Monash honours project on octonions. I also received inspiration from some of the Internet postings of John Baez, and help on technical points from Terry Gannon.

2. REGULAR POLYHEDRA. The first exceptional objects to emerge in mathematics were the five regular polyhedra, known to the Greeks and the Etruscans around 500 BC.

These are exceptional because there are *only* five of them, whereas there are infinitely many regular polygons.

The Pythagoreans may have known a proof, by considering angles, that only five regular polyhedra exist. Many more of their properties were worked out by Thaetetus, around 375 BC, and by 300 BC they were integrated into the general theory of numbers and geometry in Euclid's *Elements*. The construction of the regular polyhedra in Book XIII, and the proof that there are only five of them, is



Tetrahedron Octahedron Icosahedron Cube Dodecahedron

Figure 1

the climax of the *Elements*. And the elaborate theory of irrational magnitudes in Book X is probably motivated by the magnitudes arising from the regular polyhedra, such as $((5 + \sqrt{5})/2)^{1/2}$, the diagonal of the icosahedron with unit edge.

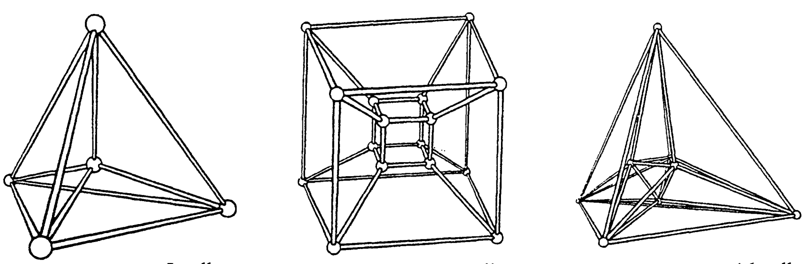
Thus it is probable that the regular polyhedra were the inspiration for the *Elements*, and hence for most of the later development of mathematics. If I wished to demonstrate the influence of exceptional objects on mathematics, I could rest my case right there. But there is much more. The most interesting cases of influence have been comparatively recent, but before discussing them we should recall a famous example from 400 years ago.

In Kepler's *Mysterium Cosmographicum* of 1596 the regular polyhedra made a spectacular, though premature, appearance in mathematical physics. Kepler "explained" the distances from the sun of the six known planets by a model of six spheres inscribing and circumscribing the five regular polyhedra. Alas, while geometry could not permit more regular polyhedra, physics could permit more planets, and the regular polyhedra were blown out of the sky by the discovery of Uranus in 1781.

Kepler of course never knew the fatal flaw in his model, and to the end of his days it was his favourite creation. Indeed it was not a complete waste of time, because it also led him to mathematical results of lasting importance about polyhedra.

3. REGULAR POLYTOPES. It is a measure of the slow progress of mathematics until recent times that the place of the regular polyhedra did not essentially change until the 19th century. Around 1850, they suddenly became part of an infinite panorama of exceptional objects, the n-dimensional analogues of polyhedra, called *polytopes*. Among other things, this led to the realisation that the dodecahedron and icosahedron are *more exceptional* than the other polyhedra, because the tetrahedron, cube, and octahedron have analogues in all dimensions.

The 4-dimensional analogues of the tetrahedron, cube, and octahedron are known as the 5-cell (because it is bounded by 5 tetrahedra), the 8-cell (bounded by 8 cubes), and the 16-cell (bounded by 16 tetrahedra), respectively.



5-cell 8-cell 16-cell

Figure 2

The other 4-dimensional regular polytopes were discovered by Schläfli in 1852: the 24-cell (bounded by 24 octahedra), 120-cell (bounded by 120 dodecahedra), and 600-cell (bounded by 600 tetrahedra).

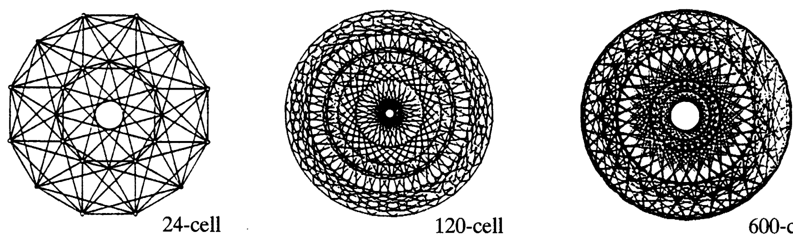


Figure 3

Schläfli also discovered that in five or more dimensions the only regular polytopes are the analogues of the tetrahedron, cube, and octahedron. Thus from the n-dimensional perspective the dodecahedron, icosahedron, 24-cell, 120-cell, and 600-cell are the genuine exceptions. This also suggests that 3 and 4 are exceptional dimensions, a fact that just might explain the dimension of the space (or spacetime) in which we live, though after Kepler one should be cautious about such speculations!

4. INFINITE FAMILIES PLUS EXCEPTIONS. Schläfli's discovery may be summarised by saying that the regular polytopes may be classified into three infinite families plus five exceptions.

The classification of other structures follows a similar pattern:

- Regular tessellations of $\mathbb{R}^n$: one infinite family and four exceptional tessellations. These were enumerated by Schläfli in 1852. The infinite family is the tessellation of $\mathbb{R}^n$ by "n-cubes", which generalises the tessellation of the plane by squares. Two of the exceptions are the dual tessellations of $\mathbb{R}^2$ by equilateral triangles and regular hexagons. The other two, discovered by Schläfli, are dual tessellations of $\mathbb{R}^4$, by 16-cells and 24-cells.
- Simple Lie groups: four infinite families A_n, B_n, C_n, D_n, plus five exceptional groups G_2, F_4, E_6, E_7, E_8. The infinite families were discovered by Lie, and the exceptions by Killing in 1888 and Cartan in 1894. The process of classification resembles that of regular polyhedra, because it reduces these continuous groups (in a far from obvious way) to discrete arrangements of line segments in Euclidean space (root systems), with certain constraints on angles and relative lengths.
- Finite reflection groups: four infinite families plus seven exceptional groups, discovered by Coxeter in 1934. These generalise the symmetry groups of polyhedra, and are linked to the simple Lie groups via root systems. It emerged from Coxeter's work, and also the work of Weyl in 1925, Cartan in 1926, and Stiefel in 1942, that each simple Lie group is determined by a reflection group, now called its *Weyl group*.
- Finite simple groups: 18 infinite families plus 26 exceptional (sporadic) groups, discovered by the collective work of many mathematicians between 1830 and 1980. These are also linked to the simple Lie groups by passage from the continuous to the discrete. Galois, around 1830, first noticed that finite simple groups arise from groups of transformations when complex

coefficients are replaced by elements of a finite field. The idea was extended by Jordan and Dickson, and reached full generality with Chevalley in 1955.

However, even the exceptional Lie groups yield infinite families of finite simple groups, so the sporadic simple groups appear to be the height of exceptionality. Still, they are not completely unrelated to the other exceptional objects. Other links between exceptions appear when we pick up other threads of the story.

5. SUMS OF SQUARES. Entirely different from the story of polyhedra, but at least as old, is the story of sums of squares. Sums of two squares have been studied since the Babylonian discovery of "Pythagorean triples" around 1800 BC. Around 200 AD, Diophantus made the striking discovery that sums of two squares can be multiplied, in a certain sense. His *Arithmetica*, Book III, Problem 19 says

> 65 is "naturally" divided into two squares in two ways... due to the fact that 65 is the product of 13 and 5, each of which numbers is the sum of two squares.

In 950 AD, al-Khazin interpreted this as a reference to the *two square identity*

$$\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right) = \left(a_1 a_2 \mp b_1 b_2\right)^2 + \left(b_1 a_2 \pm a_1 b_2\right)^2,$$

as did Fibonacci in his *Liber Quadratorum* of 1225. Fibonacci also gave a proof, which is not trivial in his algebra—it takes five pages!

There is no similar identity for sums of three squares, as Diophantus probably realised. 15 is not a sum of three integer squares, yet $15 = 5 \times 3$, and $3 = 1^2 + 1^2 + 1^2$, $5 = 0^2 + 1^2 + 2^2$. Thus there can be no identity, with integer coefficients, expressing the product of sums of three squares as a sum of three squares. It is also true that 15 is not the sum of three *rational* squares. Diophantus may have known this too. He stated that 15 is not the sum of two rational squares, and the proof for three squares is similar (involving congruence mod 8 instead of mod 4).

But claims about sums of four squares are conspicuously absent from the *Arithmetica*. This led Bachet to conjecture, in his 1621 edition of the book, that *every natural number is the sum of four (natural number) squares.*

Fermat claimed a proof of Bachet's conjecture, but the first documented step towards a proof was Euler's *four square identity*, given in a letter to Goldbach, 4 May 1748:

$$\begin{aligned}
&\left(a^2 + b^2 + c^2 + d^2\right)\left(p^2 + q^2 + r^2 + s^2\right) \\
&\qquad = \left(ap + bq + cr + ds\right)^2 + \left(aq - bp - cs + dr\right)^2 \\
&\qquad\qquad + \left(ar + bs - cp - dq\right)^2 + \left(as - br + cq - dp\right)^2.
\end{aligned}$$

Using this, Lagrange completed the proof of Bachet's conjecture in 1770.

In 1818, Degen discovered an *eight square identity*, which turned out to be the last in the series, so sums of 2, 4, and 8 squares are exceptional. This was not proved until 1898, by Hurwitz. In fact, Degen's identity was virtually unknown until the discovery of...

6. THE DIVISION ALGEBRAS $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. A division algebra of dimension n over $\mathbb{R}$ consists of n-tuples of real numbers under vector addition, together with a "multiplication" that distributes over addition and admits "division". The idea is to make n-tuples add and multiply as "n-dimensional numbers". Addition is no

problem, but a decent multiplication is exceptional—apart from the obvious case $n = 1$ it exists only in dimensions 2, 4, and 8. The first clue how to multiply in each of these dimensions was the identity for sums of 2, 4, or 8 squares.

Diophantus associated $a^2 + b^2$ with the pair (a, b), identified with the right-angled triangle with sides a and b. From triangles (a_1, b_1) and (a_2, b_2) he formed the "product" triangle $(a_1a_2 - b_1b_2, b_1a_2 + a_1b_2)$. His identity

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2) = (a_1a_2 - b_1b_2)^2 + (b_1a_2 + a_1b_2)^2$$

shows that the hypotenuse of the "product" is the product of the hypotenuses. In modern notation his identity is

$$|z_1|^2|z_2|^2 = |z_1z_2|^2 \quad \text{where} \quad z_1 = a_1 + ib_1, \quad z_2 = a_2 + ib_2,$$

and it expresses the *multiplicative property of the norm* $|z|^2 = a^2 + b^2$ of $z = a + ib$: the norm of a product is the product of the norms. A multiplicative norm makes division possible, because it guarantees that the product of nonzero elements is nonzero.

The same product of pairs (without the identification with triangles) was rediscovered by Hamilton in 1835 as a *definition* of the product of complex numbers:

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 - b_1b_2, b_1a_2 + a_1b_2).$$

Hamilton had been trying since 1830 to define multiplication of n-tuples and retain the basic properties of multiplication on $\mathbb{R}$ and $\mathbb{C}$:

- multiplication is commutative and associative
- multiplication is distributive over vector addition
- the norm $x_1^2 + x_2^2 + \cdots + x_n^2$ of $(x_1, x_2, \ldots, x_n)$ is multiplicative.

He got stuck on triples for 13 years, not knowing that a multiplicative norm was ruled out by elementary results on sums of three squares.

If he had known this earlier, would he have given up the whole idea, without trying quadruples? van der Waerden [7, p. 185] thought so, implying that Hamilton tried quadruples in October 1843 only to salvage something from his long and fruitless commitment to triples. He had already given up on commutative multiplication, but with quadruples he saved the other properties, in his algebra $\mathbb{H}$ of *quaternions*. It was lucky, in van der Waerden's opinion, that Hamilton didn't know about sums of three squares.

But what if he had known about sums of four squares? He would then have seen a multiplicative norm of quadruples, and would perhaps have discovered quaternions in 1830. This is not pure speculation. Gauss, who knew Euler's four square identity, discovered both a "quaternion form" of it and also a "quaternion representation" of rotations of the sphere, the latter around 1819.

However, Gauss did not publish these discoveries, so Hamilton was lucky after all—he was first to see the full structure of the quaternions, and he received all the credit for them.

John Graves, who had been in correspondence with Hamilton for years on the problem of multiplying n-tuples, was galvanised by the discovery of quaternions and the associated four square identity (which Hamilton and he at that time believed to be new). In December 1843 Graves rediscovered Degen's eight square identity, and immediately constructed the 8-dimensional division algebra $\mathbb{O}$ of *octonions*.

Dickson [4, p. 159] condensed all these results into one formula, showing that each algebra in the sequence $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ comes from the one before by a simple generalisation of Diophantus' rule for multiplying pairs:

$$(a_1, b_1)(a_2, b_2) = \left(a_1a_2 - \overline{b_2}b_1, b_2a_1 + b_1\overline{a_2}\right) \quad \text{where} \quad \overline{(a, b)} = (\bar{a}, -b).$$

From this it is easily proved that the quaternions are associative but not commutative, and that the octonions are not associative. The next algebra in the sequence, consisting of pairs (a, b) of octonions, is not a division algebra. Theorems by Frobenius, Zorn, and others confirm that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are indeed exceptional—they are the only finite-dimensional division algebras over $\mathbb{R}$. For further information on them see [5].

7. LATTICES. The polyhedron thread of our story intertwines with the division algebra thread when we reconsider tessellations of $\mathbb{R}^n$. The two exceptional tessellations of $\mathbb{R}^2 = \mathbb{C}$, by equilateral triangles and hexagons, are both based on the lattice of *Eisenstein integers*

$$m + n\frac{1 + \sqrt{-3}}{2} \quad \text{for} \quad m, n \in \mathbb{Z}.$$

The triangle tessellation has lattice points at vertices; the hexagon tessellation has them at face centres.

Similarly, the two exceptional tessellations of $\mathbb{R}^4 = \mathbb{H}$ are based on the *Hurwitz integers*

$$p\frac{\mathbf{1} + \mathbf{i} + \mathbf{j} + \mathbf{k}}{2} + q\mathbf{i} + r\mathbf{j} + s\mathbf{k} \quad \text{for} \quad p, q, r, s \in \mathbb{Z},$$

called the "integer quaternions," and were used by Hurwitz in 1896 to give a new proof that every natural number is a sum of four squares.

The analogous "integer octonions" form a lattice in $\mathbb{R}^8$, in which the neighbours of each lattice point form a polytope discovered by Gosset in 1897. It is not regular, but is nevertheless highly symmetrical. Its symmetry group is none other than the Weyl group of the exceptional Lie group E_8. Gosset's story, and the history of regular polytopes in general, is told in [3].

8. PROJECTIVE CONFIGURATIONS. Other exceptional structures in geometry are the projective configurations discovered by Pappus, around 300 AD, and by Desargues in 1639.

Theorem of Pappus. *If the vertices of a hexagon lie alternately on two straight lines, then the intersections of the opposite sides of the hexagon lie in a straight line.*

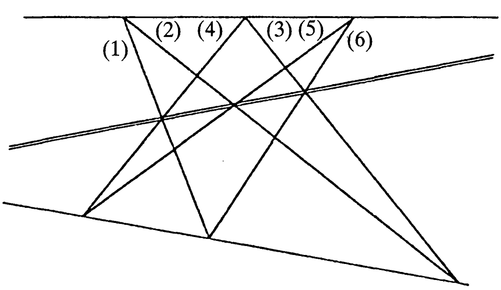


Figure 4

Theorem of Desargues. *If two triangles are in perspective, then the intersections of their corresponding sides lie in a line.*

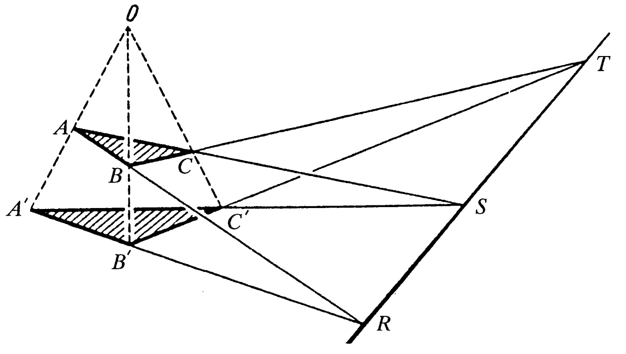


Figure 5

These statements are "projective" because they involve only *incidence*: whether or not points meet lines. Moreover, Desargues' theorem in space has a *proof* from incidence properties—the properties that

- two planes meet in a line,
- two lines in the same plane meet in a point.

The diagram of the Desargues configuration hints at this, by suggesting that the triangles lie in three dimensions, even though the diagram itself necessarily lies in the plane. Yet, strangely, the Desargues and Pappus theorems in the plane do *not* have proofs from obvious incidence properties; their proofs involve the concept of distance or coordinates.

Why is this so? In 1847 von Staudt gave geometric constructions of $+$ and $\times$, thus "coordinatising" each projective plane by a division ring. Then in 1899 Hilbert made the wonderful discovery that the geometry of the plane is tied to the algebra of the ring:

- *Pappus' theorem holds* $\Leftrightarrow$ *the division ring is commutative*
- *Desargues' theorem holds* $\Leftrightarrow$ *the division ring is associative.*

Conversely, any division ring R yields a projective plane RP^2. So by Hilbert's theorem,

- $\mathbb{R}P^2$ and $\mathbb{C}P^2$ satisfy Pappus,
- $\mathbb{H}P^2$ satisfies Desargues but not Pappus, and
- $\mathbb{O}P^2$ satisfies neither.

In 1933 Ruth Moufang completed Hilbert's results with a theorem satisfied by $\mathbb{O}P^2$, the "little Desargues' theorem", which states uniqueness of the construction of the fourth harmonic point D of points A, B, C. She showed that "little

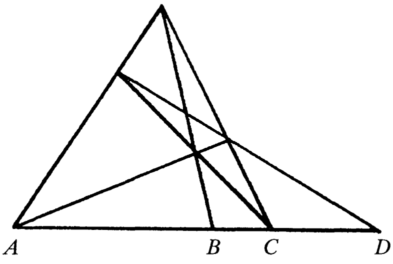


Figure 6

Desargues" holds if and only if the division ring is *alternative*, that is, for all a and b, $a(ab) = (aa)b$ and $b(aa) = (ba)a$.

Alternativity is a characteristic property of $\mathbb{O}$ by the following result of Zorn from 1930: a finite-dimensional alternative division algebra over $\mathbb{R}$ is $\cong \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. Thus the octonion projective plane $\mathbb{O}P^2$ is exceptional, simply because $\mathbb{O}$ is. However, $\mathbb{O}P^2$ is *more* exceptional than $\mathbb{R}P^2$, $\mathbb{C}P^2$, and $\mathbb{H}P^2$, because each of the latter belongs to an infinite family of projective spaces.

The method of homogeneous coordinates can be used to construct a projective space RP^n of each dimension n for $R = \mathbb{R}, \mathbb{C}, \mathbb{H}$. But there is no $\mathbb{O}P^3$, because

$$\text{existence of } \mathbb{O}P^3 \Rightarrow \text{Desargues' theorem holds} \Rightarrow \mathbb{O} \text{ is associative.}$$

Thus $\mathbb{O}P^1$ and $\mathbb{O}P^2$ are *exceptional projective spaces*, compared with the infinite families $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$.

9. $\mathbb{O}$: THE MOTHER OF ALL EXCEPTIONS?. We have already observed that many objects inherit their classification from that of the simple Lie groups A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8. In particular, classifications based on the A, D, and E series arise so often that explaining them has become a flourishing industry. For a recent survey of this field of "ADE classifications" see **[6]**.

To find the source of such classifications, we should try to understand where the simple Lie groups come from. The groups A_n, B_n, C_n, D_n are automorphism groups of n-dimensional projective spaces (or subgroups), so the infinite families of simple Lie groups arise from the infinite families of projective spaces $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$.

Since there is no $\mathbb{O}P^n$ for $n > 2$, one does not expect many Lie groups to come from the octonions, but in fact all five exceptional Lie groups are related to $\mathbb{O}$. It seems that, in some sense, the exceptional Lie groups inherit their exceptionality from $\mathbb{O}$.[1]

This astonishing relationship began to emerge when Cartan discovered that $G_2 \cong \mathrm{Aut}(\mathbb{O})$. He mentioned this casually in a 1908 article **[2]** on the history of hypercomplex number systems, saying only that the automorphism group of the octonions is a simple Lie group with 14 parameters. However, he knew from his

[1]The five exceptional Lie groups could be of truly cosmic importance. Recent work in theoretical physics seeks to reconcile general relativity with quantum theory by means of "string theories" in which atomic particles arise as modes of vibration. The possible string theories correspond to the exceptional Lie groups, hence string theory allows five "possible worlds". This point was raised by Ed Witten in his Gibbs Lecture at the AMS-MAA Joint Meetings in Baltimore 1998, along with the question: if there are five possible worlds, who lives in the other four?

1894 classification of the simple Lie groups that G_2 is the only compact simple Lie group of dimension 14.

In the 1950s several constructions of F_4, E_6, E_7, and E_8 from $\mathbb{O}$ were discovered by Freudenthal, Tits, and Rosenfeld. For example, they found that F_4 is the isometry group of $\mathbb{O}P^2$, and E_6 is its collineation group. Their success in finding links between these exceptional objects raises an interesting question, though perhaps one that will never be completely answered: How many exceptional objects inherit their exceptionality from $\mathbb{O}$?

Some connections between $\mathbb{O}$ and sporadic simple groups are known, but the latter groups remain the most mysterious exceptions to date. Naturally, the sheer mystery of these groups has only intensified the search for a broad theory of exceptional objects. This has led to some surprising developments, among them a revival of the near-dead subject of "foundations of geometry". Foundations have now grown into "buildings" (a concept due to Tits), which are just part of a huge field of "incidence geometry" spanning most of the results we have discussed. A comprehensive survey of this new field with ancient roots may be found in **[1]**.

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The Chauvenet Prize for Mathematical Exposition has been awarded to Professor P. R. Halmos of the Institute for Advanced Study for his paper entitled "The Foundations of Probability," published in this MONTHLY for November, 1944. This most recent award of the Prize "for a noteworthy expository paper published in English by a member of the Association" covers the three-year period, 1944–'46.

The Association first established the Chauvenet Prize in 1925. At that time it was specified that the award was to be made every five years for the best article of an expository character dealing with some mathematical topic, written by a member of the Association and published in English during the five calendar years preceding the award. The Prize was not to be awarded for books. Originally the amount of the award was fixed at one hundred dollars.

At a later date it was decided to award the Prize every three years, and the amount was changed to fifty dollars. In 1942, it was further specified that only such papers would be considered as "came within the range of profitable reading of Association members."

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